

# A SLIGHT GENERALIZATION OF KELLER'S THEOREM

VERED MOSKOWICZ

**ABSTRACT.** The famous Jacobian problem asks: Is a morphism  $f : \mathbb{C}[x, y] \rightarrow \mathbb{C}[x, y]$  having an invertible Jacobian, invertible? If we add the assumption that  $\mathbb{C}(f(x), f(y)) = \mathbb{C}(x, y)$ , then  $f$  is invertible; this result is due to O. H. Keller (1939). We suggest the following slight generalization of Keller's theorem: If  $f : \mathbb{C}[x, y] \rightarrow \mathbb{C}[x, y]$  is a morphism having an invertible Jacobian, and if there exist  $n \geq 1$ ,  $a \in \mathbb{C}(f(x), f(y))^*$  and  $b \in \mathbb{C}(f(x), f(y))$  such that  $(ax + b)^n \in \mathbb{C}(f(x), f(y))$ , then  $f$  is invertible. A similar result holds for  $\mathbb{C}[x_1, \dots, x_m]$ .

## 1 Introduction

Throughout this paper we work over  $\mathbb{C}$  (Theorems 3.1 and 3.2) or just over a field of characteristic zero (Theorems 4.1, 4.3, 5.1 and 5.1). Some of the known results that we recall in the preliminaries section are true over a PID or a UFD or even any ring; sometimes  $2 \neq 0$ . However, in view of [6, Lemma 1.1.14], it seems that  $\mathbb{C}$  is good enough. Notice that in our main theorem, Theorem 3.1, we apply Formanek's theorem [7, Theorem 2] which is over  $\mathbb{C}$ . Even if Formanek's theorem is still valid over a more general ring,  $\mathbb{R}$  is not enough for our main theorem, since we wish to have all  $n$ 'th roots of unity for some  $n$  which is probably  $\geq 3$ .

In all our theorems:  $k = \mathbb{C}$  or  $k$  is a field of characteristic zero,  $f : k[x, y] \rightarrow k[x, y]$  is a morphism that satisfies  $\text{Jac}(f(x), f(y)) \in k^*$ , and  $p := f(x)$ ,  $q := f(y)$ . More generally, when  $m \geq 2$ :  $f : k[x_1, \dots, x_m] \rightarrow k[x_1, \dots, x_m]$  is a morphism that satisfies  $\text{Jac}(f(x_1), \dots, f(x_m)) \in k^*$ , and  $r_1 := f(x_1), \dots, r_m := f(x_m)$ . Keller's theorem says that the birational case has a positive answer: If  $\mathbb{C}(r_1, \dots, r_m) = \mathbb{C}(x_1, \dots, x_m)$ , then  $f$  is invertible. It can be found in Keller's paper [9], van den Essen's book [6, Corollary 1.1.35] and Bass-Connell-Wright's paper [3, Theorem 2.1 (a) iff (b)]. When  $m = 2$ , Keller's theorem says that if  $\mathbb{C}(p, q) = \mathbb{C}(x, y)$ , then  $f$  is invertible. We bring our observation that it is possible to slightly generalize Keller's theorem: Instead of demanding  $\mathbb{C}(x, y) = \mathbb{C}(p, q)$ , it is enough to demand that  $\mathbb{C}(p, q)((ax + b)^n) = \mathbb{C}(p, q)$ , for some  $n \geq 1$ ,  $a \in \mathbb{C}(p, q)^*$  and  $b \in \mathbb{C}(p, q)$ . The general case when  $m \geq 2$  also seems to be true, namely, instead of demanding  $\mathbb{C}(x_1, \dots, x_m) = \mathbb{C}(r_1, \dots, r_m)$ , it is enough to demand that for  $m - 1$  variables  $x_j$   $\mathbb{C}(r_1, \dots, r_m)((a_j x_j + b_j)^{n_j}) = \mathbb{C}(r_1, \dots, r_m)$ , for some  $n_j \geq 1$ ,  $a_j \in \mathbb{C}(r_1, \dots, r_m)^*$  and  $b_j \in \mathbb{C}(r_1, \dots, r_m)$ .

*Remark 1.1.* Of course, instead of taking  $\mathbb{C}(p, q)((ax + b)^n)$ , we could have taken  $\mathbb{C}(p, q)((ay + b)^n)$ . The general case  $m \geq 2$  is already in its general form; namely, conditions on any  $m - 1$  variables from  $x_1, \dots, x_m$ , not necessarily the first  $m - 1$  variables.

---

2010 *Mathematics Subject Classification.* Primary 14R15, 16W20.

The author was partially supported by an Israel-US BSF grant 2010/149.

## 2 Preliminaries

We rely on the following nice results:

**(1) Wang's and Bass's theorem:** Let  $k$  be a UFD. Then:  $k[x_1, \dots, x_m] \cap k(r_1, \dots, r_m) = k[r_1, \dots, r_m]$ . This result was proved by Wang [13, Theorem 41(i)] and generalized by Bass for not necessarily polynomial rings [4, Remark after Corollary 1.3, page 74] See also [6, Corollary 1.1.34 and Proposition D.1.7]. When  $m = 2$ :  $k[x, y] \cap k(p, q) = k[p, q]$ .

**(2) Equivalent special cases, each having a positive answer to the Jacobian Conjecture:** Let  $k$  be a field of characteristic zero. Then TFAE:

- The birational case:  $k(x_1, \dots, x_m) = k(r_1, \dots, r_m)$ .
- The Galois case:  $k(x_1, \dots, x_m)$  is a Galois extension of  $k(r_1, \dots, r_m)$ .
- The integral case:  $k[x_1, \dots, x_m]$  is integral over  $k[r_1, \dots, r_m]$ .
- The projective case:  $k[x_1, \dots, x_m]$  is a projective  $k[r_1, \dots, r_m]$ -module.
- $f$  is invertible.

This and other equivalent conditions can be found in [13, Theorem 46(iii)] and in [3, Theorem 2.1]. Some of the above equivalent conditions can be found in [9] (Keller's paper from 1939), [5], [1], [11], [15], [10], [14, Theorem 9], [6, Theorem 2.2.16] and probably in other places as well.

When  $m = 2$ , TFAE:  $k(x, y) = k(p, q)$ ;  $k(x, y)$  is a Galois extension of  $k(p, q)$ ;  $k[x, y]$  is integral over  $k[p, q]$ ;  $k[x, y]$  is a projective  $k[p, q]$ -module.

**(3) Formanek's theorem:**  $\mathbb{C}(r_1, \dots, r_m, x_1, \dots, x_{m-1}) = \mathbb{C}(x_1, \dots, x_m)$ . See [7, Theorem 2]. When  $m = 2$ :  $\mathbb{C}(p, q, x) = \mathbb{C}(x, y)$ . (Of course, we also have  $\mathbb{C}(p, q, y) = \mathbb{C}(x, y)$ . Similarly, the equality in the general case is true adjoining any  $m - 1$  variables).

**(4) Hamann's theorem:** If  $p$  and  $q$  are monic in  $y$ , then  $k[x, y]$  is integral over  $k[p, q][x]$ . (having the same field of fractions:  $k(x, y) = k(p, q)(x)$ ). See [8, Proposition 2.1]. If  $p$  and  $q$  of a given morphism  $f$  are not monic in  $y$ , then one can consider  $gf$  for an appropriate automorphism  $g$ , and get  $\tilde{p} := (gf)(x) = g(p)$  and  $\tilde{q} := (gf)(y) = g(q)$  monic in  $y$ , see [8, Theorem 1.1] ("change of variables").

Although not mentioned explicitly in Hamann's paper, it seems that the following is also true: If  $r_1, \dots, r_m$  are monic in  $x_m$ , then  $k[x_1, \dots, x_m]$  is integral over  $k[r_1, \dots, r_m][x_1, \dots, x_{m-1}]$ .

*Remark 2.1.* An equivalent way to say  $k(x, y) = k(p, q)$  is:  $x \in k(p, q)$  and  $y \in k(p, q)$ . Actually, it is enough to demand that  $x \in k(p, q)$ : Assume  $x \in k(p, q)$ . By Formanek's theorem  $k(p, q, x) = k(x, y)$ . Combining the two yields  $k(p, q) = k(x, y)$ .

Similarly for the general case  $m \geq 2$ : It is enough to demand that  $x_j \in k(r_1, \dots, r_m)$  for  $1 \leq j \leq m - 1$  (namely, omitting the demand  $x_m \in k(r_1, \dots, r_m)$ ): By Formanek's theorem  $k(r_1, \dots, r_m, x_1, \dots, x_{m-1}) = k(x_1, \dots, x_m)$ , and combining this with our  $m - 1$  conditions yields  $k(r_1, \dots, r_m) = k(x_1, \dots, x_m)$ .

## 3 The slight generalization

After the above preparations it is time to prove our main theorem:

**Theorem 3.1.** *If there exist  $n \geq 1$ ,  $a \in \mathbb{C}(p, q)^*$  and  $b \in \mathbb{C}(p, q)$  such that  $(ax + b)^n \in \mathbb{C}(p, q)$ , then  $f$  is invertible*

Recall that  $\mathbb{C}(p, q) \subseteq \mathbb{C}(x, y)$  is finite ( $x$  and  $y$  are algebraic over  $\mathbb{C}(p, q)$ ), since  $\{p, q, x\}$  and  $\{p, q, y\}$  are algebraically dependent over  $\mathbb{C}$ . See [6, Proposition 1.1.31]) and separable.

*Proof.* Denote  $A = \mathbb{C}(p, q)$  and  $B = \mathbb{C}(x, y)$ .

- $\mathbb{C}(p, q, (ax + b)^n) = A$  by our assumption.
- $\mathbb{C}(p, q, ax + b) = \mathbb{C}(p, q, x) = B$ ; the first equality is trivial, and the second equality is Formanek's theorem mentioned above

We show that  $A \subseteq B$  is a Galois extension or  $f$  is birational (which also means that  $A \subseteq B$  is Galois, trivially), hence  $f$  is invertible, as was recalled in the preliminaries section.

Let  $h := T^n - (ax + b)^n \in \mathbb{C}(p, q, (ax + b)^n)[T]$ .

There are two options:

**First option:**  $h$  is separable. So  $\mathbb{C}(p, q, ax + b)$  is the splitting field of the separable polynomial  $T^n - (ax + b)^n \in \mathbb{C}(p, q, (ax + b)^n)[T]$ . Therefore  $A \subseteq B$  is Galois (see [12, Definition 4.47]).

**Second option:**  $h$  is not separable. Hence  $\epsilon_u(ax + b) = \epsilon_v(ax + b)$ , where  $\epsilon_u \neq \epsilon_v$  are two different  $n$ 'th roots of 1. Then from  $(\epsilon_u - \epsilon_v)(ax + b) = 0$ , we get that  $ax + b = 0$ . So  $x = -b/a \in \mathbb{C}(p, q)$ , and we get  $\mathbb{C}(p, q) = \mathbb{C}(p, q, ax + b) = \mathbb{C}(p, q, x) = \mathbb{C}(x, y)$ . Therefore we arrived at the birational case.  $\square$

It seems that a similar result holds for  $\mathbb{C}[x_1, \dots, x_m]$ :

**Theorem 3.2.** *If there exist  $n_1, \dots, n_{m-1} \geq 1$ ,  $a_1, \dots, a_{m-1} \in \mathbb{C}(r_1, \dots, r_m)^*$  and  $b_1, \dots, b_{m-1} \in \mathbb{C}(r_1, \dots, r_m)$  such that for each  $1 \leq j \leq m-1$   $(a_j x_j + b_j)^{n_j} \in \mathbb{C}(r_1, \dots, r_m)$ , then  $f$  is invertible.*

Recall that  $\mathbb{C}(r_1, \dots, r_m) \subseteq \mathbb{C}(x_1, \dots, x_m)$  is finite (for every  $1 \leq i \leq m$ ,  $x_i$  is algebraic over  $\mathbb{C}(r_1, \dots, r_m)$ , since  $\{r_1, \dots, r_m, x_i\}$  are algebraically dependent over  $\mathbb{C}$ . See [6, Proposition 1.1.31]) and separable.

*Proof.* Denote  $A = \mathbb{C}(r_1, \dots, r_m)$  and  $B = \mathbb{C}(x_1, \dots, x_m)$ .

- $\mathbb{C}(r_1, \dots, r_m, (a_1 x_1 + b_1)^{n_1}, \dots, (a_{m-1} x_{m-1} + b_{m-1})^{n_{m-1}}) = A$  by our assumptions.
- $\mathbb{C}(r_1, \dots, r_m, a_1 x_1 + b_1, \dots, a_{m-1} x_{m-1} + b_{m-1}) = \mathbb{C}(r_1, \dots, r_m, x_1, \dots, x_{m-1}) = B$ ; the first equality is trivial, and the second equality is Formanek's theorem mentioned above

We show that  $A \subseteq B$  is a Galois extension or  $f$  is birational, hence  $f$  is invertible.

Let  $h := \prod_j h_j \in \mathbb{C}(r_1, \dots, r_m, (a_1 x_1 + b_1)^{n_1}, \dots, (a_{m-1} x_{m-1} + b_{m-1})^{n_{m-1}})[T]$ , where  $h_j := T^{n_j} - (a_j x_j + b_j)^{n_j}$ ,  $1 \leq j \leq m-1$ .

There are two options:

**First option:**  $h$  is separable. So  $\mathbb{C}(r_1, \dots, r_m, a_1 x_1 + b_1, \dots, a_{m-1} x_{m-1} + b_{m-1})$  is the splitting field of the separable polynomial  $h \in \mathbb{C}(r_1, \dots, r_m, (a_1 x_1 + b_1)^{n_1}, \dots, (a_{m-1} x_{m-1} + b_{m-1})^{n_{m-1}})[T]$ . Therefore  $A \subseteq B$  is Galois.

**Second option:**  $h$  is not separable. Multiple roots of  $h$  can be of the following two forms:

- $h_{j_0}$  is not separable: There exists  $1 \leq j_0 \leq m-1$  such that  $\epsilon_u(a_{j_0} x_{j_0} + b_{j_0}) = \epsilon_v(a_{j_0} x_{j_0} + b_{j_0})$ , where  $\epsilon_u \neq \epsilon_v$  are two different  $n_{j_0}$ 'th roots of 1. Then from  $(\epsilon_u - \epsilon_v)(a_{j_0} x_{j_0} + b_{j_0}) = 0$ , we get that  $a_{j_0} x_{j_0} + b_{j_0} = 0$ . So  $x_{j_0} = -b_{j_0}/a_{j_0} \in \mathbb{C}(r_1, \dots, r_m)$ .
- $h_{i_0}$  and  $h_{j_0}$  are not relatively prime: There exist  $1 \leq i_0 \neq j_0 \leq m-1$  such that  $\epsilon(a_{i_0} x_{i_0} + b_{i_0}) = \delta(a_{j_0} x_{j_0} + b_{j_0})$ , where  $\epsilon^{n_{i_0}} = 1$  and  $\delta^{n_{j_0}} = 1$ . Then  $x_{j_0} \in \mathbb{C}(r_1, \dots, r_m, x_{i_0})$ .

In each form we get that  $x_{j_0}$  is a redundant generator of

$$\mathbb{C}(r_1, \dots, r_m, x_1, \dots, x_{m-1}) = \mathbb{C}(r_1, \dots, r_m, a_1 x_1 + b_1, \dots, a_{m-1} x_{m-1} + b_{m-1}).$$

After removing all the redundant  $x_j$ 's, we get one of the two following options:

- $\mathbb{C}(r_1, \dots, r_m, x_1, \dots, x_{m-1}) = \mathbb{C}(r_1, \dots, r_m, x_{t_1}, \dots, x_{t_s})$ , where  $1 \leq t_1 < \dots < t_s \leq m-1$  are such that  $h_{t_1}, \dots, h_{t_s}$  are separable and relatively

prime Then  $B = \mathbb{C}(r_1, \dots, r_m, a_{t_1}x_{t_1} + b_{t_1}, \dots, a_{t_s}x_{t_s} + b_{t_s})$  is the splitting field of the separable polynomial  $h_{t_1} \cdots h_{t_s} \in A[T]$ , so  $A \subseteq B$  is Galois.

- There are no such  $h_{t_1}, \dots, h_{t_s}$ . Then for each  $1 \leq j \leq m-1$ ,  $x_j \in \mathbb{C}(r_1, \dots, r_m)$ , hence  $A = B$ , the birational case.

□

Inspired by [13, Theorem 46], [3, Theorem 2.1] and [6, Theorem 2.2.16], we wish to ask the following question: Is it possible to find a weaker condition than  $\mathbb{C}(p, q) \subseteq \mathbb{C}(x, y)$  being a Galois extension which is equivalent to  $\mathbb{C}(p, q)(ax + b)^n = \mathbb{C}(p, q)$ ? Since  $\mathbb{C}(p, q) \subseteq \mathbb{C}(x, y)$  is finite and separable, the weaker condition should be weaker than  $\mathbb{C}(p, q) \subseteq \mathbb{C}(x, y)$  being normal.

Also: Is it possible to find a weaker condition than integrality/projectivity of  $\mathbb{C}[x, y]$  over  $\mathbb{C}[p, q]$  which is equivalent to  $\mathbb{C}(p, q)(ax + b)^n = \mathbb{C}(p, q)$ ? Recall that  $\text{pd}_{\mathbb{C}[p, q]}(\mathbb{C}[x, y]) \in \{0, 1\}$  [13, Theorem 44(i)], and  $\text{pd}_{\mathbb{C}[p, q]}(\mathbb{C}[x, y]) = 0$  if and only if  $f$  is invertible [13, Theorem 46(i)+(iii) (1) iff (9)]. Hence, if one can show that  $\text{pd}_{\mathbb{C}[p, q]}(\mathbb{C}[x, y]) = 1$  implies  $\mathbb{C}(p, q)(ax + b)^n = \mathbb{C}(p, q)$ , then the Jacobian Conjecture is true. (Since  $\text{pd}_{\mathbb{C}[p, q]}(\mathbb{C}[x, y]) \in \{0, 1\}$  and each of the two cases implies that  $f$  is invertible).

## 4 A special case with another proof

Now we consider a special case in which the following two conditions are assumed:

- (1)  $r_1, \dots, r_m$  are monic in  $x_m$ .
- (2) Instead of  $a_1, \dots, a_{m-1} \in \mathbb{C}(r_1, \dots, r_m)^*$ ,  $b_1, \dots, b_{m-1} \in \mathbb{C}(r_1, \dots, r_m)$ , assume  $a_1, \dots, a_{m-1} \in \mathbb{C}^*$ ,  $b_1, \dots, b_{m-1} \in \mathbb{C}[r_1, \dots, r_m]$ .

On the one hand, this section can be omitted, since we already have our above results which are valid whether  $r_1, \dots, r_m$  are monic in  $x_m$  or not, and also in our above results there are more options for  $a_j, b_j$ . On the other hand, we added this section since we wished to apply Wang's theorem [13, Theorem 41(i)] which is true over a more general ring than  $\mathbb{C}$ , and we wished to apply Hamann's theorem which is true over a field of characteristic zero. (While, as already mentioned in the introduction, for the proof of our main theorem 3.1 we better take  $\mathbb{C}$ ). Also, a somewhat similar result, Theorem 5.1 (and Theorem 5.2), is proved using Wang's and Hamann's theorems, and we do not know if it can be proved without them.

**Theorem 4.1.** *Let  $k$  be a field of characteristic zero. Assume  $p$  and  $q$  are monic in  $y$ . If there exist  $n \geq 1$ ,  $a \in k^*$  and  $b \in k[p, q]$  such that  $(ax + b)^n \in k(p, q)$ , then  $f$  is invertible.*

*Remark 4.2.* Of course, we could have assumed that  $p$  and  $q$  are monic in  $x$  and demand that  $(ay + b)^n \in k(p, q)$ , instead of  $(ax + b)^n \in k(p, q)$ .

*Proof.* Assume  $(ax + b)^n \in k(p, q)$  for some  $n \geq 1$ ,  $a \in k^*$  and  $b \in k[p, q]$ . Expand  $(ax + b)^n$  and get that  $a^n x^n + na^{n-1}x^{n-1}b + \dots + b^n \in k(p, q)$ . Hence,  $a^n x^n + na^{n-1}x^{n-1}b + \dots + b^n = c/d$ , for some  $c, d \in k[p, q]$  with  $d \neq 0$ . Multiply by  $a^{-n}$  and get that  $x^n + na^{-1}x^{n-1}b + \dots + a^{-n}b^n = a^{-n}c/d := w$ .  $w = x^n + na^{-1}x^{n-1}b + \dots + a^{-n}b^n \in k[x, y]$ , and  $w = a^{-n}c/d \in k(p, q)$ , so,  $w \in k[x, y] \cap k(p, q)$ . But  $k[x, y] \cap k(p, q) \subseteq k[p, q]$  (this is the non-trivial inclusion of  $k[x, y] \cap k(p, q) = k[p, q]$ ), hence  $w \in k[p, q]$ . Namely,  $x^n + na^{-1}x^{n-1}b + \dots + a^{-n}b^n \in k[p, q]$ , which shows that  $x$  is integral over  $k[p, q]$ . Therefore (by [2, Corollary 5.2] and the remark in that page: "finitely generated algebra + integral = finitely generated module"),  $k[p, q] \subseteq k[p, q][x]$  is integral. Combine this with Hamann's theorem that  $k[p, q][x] \subseteq k[x, y]$  is integral, and get that  $k[p, q] \subseteq k[x, y]$  is integral. Finally, since the integral case

has a positive answer (as was recalled in the preliminaries section), we get that  $f$  is invertible.  $\square$

And similarly:

**Theorem 4.3.** *Let  $k$  be a field of characteristic zero. Assume  $r_1, \dots, r_m$  are monic in  $x_m$ . If there exist  $n_1, \dots, n_{m-1} \geq 1$ ,  $a_1, \dots, a_{m-1} \in k^*$  and  $b_1, \dots, b_{m-1} \in k[r_1, \dots, r_m]$  such that for each  $1 \leq j \leq m-1$ ,  $(a_j x_j + b_j)^{n_j} \in k(r_1, \dots, r_m)$ , then  $f$  is invertible.*

*Proof.* Expand  $(a_j x_j + b_j)^{n_j}$  and get that  $(a_j)^{n_j} (x_j)^{n_j} + (n_j)(a_j)^{n_j-1} (x_j)^{n_j-1} b_j + \dots + (b_j)^{n_j} \in k(r_1, \dots, r_m)$ . Hence,  $(a_j)^{n_j} (x_j)^{n_j} + (n_j)(a_j)^{n_j-1} (x_j)^{n_j-1} b_j + \dots + (b_j)^{n_j} = c_j/d_j$ , for some  $c_j, d_j \in k[r_1, \dots, r_m]$  with  $d_j \neq 0$ . Multiply by  $(a_j)^{-n_j}$  and get (after same considerations as in the proof of Theorem 4.1) that  $x_j$  is integral over  $k[r_1, \dots, r_m]$ . Therefore,  $k[r_1, \dots, r_m] \subseteq k[r_1, \dots, r_m][x_1, \dots, x_{m-1}]$  is integral. Combine this with Hamann's theorem that  $k[r_1, \dots, r_m][x_1, \dots, x_{m-1}] \subseteq k[x_1, \dots, x_m]$  is integral, and get that  $k[r_1, \dots, r_m] \subseteq k[x_1, \dots, x_m]$  is integral. Finally, since the integral case has a positive answer, we get that  $f$  is invertible.  $\square$

*Remark 4.4.* Without knowing Formanek's theorem [7, Theorem 2], and Hamann's theorem [8, Proposition 2.1], and only knowing Wang's theorem, we could still have similar results. More precisely, instead of demanding one condition ( $m-1$  conditions), we should have demanded two conditions ( $m$  conditions), because we need to guarantee that each of the  $m$  variables  $x_1, \dots, x_m$  is integral over  $k[r_1, \dots, r_m]$ . More precisely: If there exist  $n, \tilde{n} \geq 1$ ,  $a, \tilde{a} \in k^*$  and  $b, \tilde{b} \in k[p, q]$  such that  $(ax + b)^n \in k(p, q)$  and  $(\tilde{a}y + \tilde{b})^{\tilde{n}} \in k(p, q)$ , then  $f$  is invertible. The proof is not difficult: The first condition implies that  $x$  is integral over  $k[p, q]$ , and the second condition implies that  $y$  is integral over  $k[p, q]$ . Therefore,  $k[x, y] = k[p, q][x, y]$  is integral over  $k[p, q]$ . Hence  $f$  is invertible. The general case  $m \geq 2$  is similar.

## 5 An additional result

The following result is proved quite similarly to Theorem 4.1.

**Theorem 5.1.** *Let  $k$  be a field of characteristic zero. Assume  $p$  and  $q$  are monic in  $y$ . If there exist  $n_1 > n_2 > \dots > n_l \geq 0$  and  $a_{n_2}, \dots, a_{n_l} \in k[p, q]$  such that  $x^{n_1} + a_{n_2} x^{n_2} + \dots + a_{n_l} x^{n_l} \in k(p, q)$ , then  $f$  is invertible.*

*Proof.*  $x^{n_1} + a_{n_2} x^{n_2} + \dots + a_{n_l} x^{n_l} \in k(p, q) \cap k[x, y] = k[p, q]$ . Hence,  $x$  is integral over  $k[p, q]$ . Therefore,  $k[p, q] \subseteq k[p, q][x]$  is integral. Combine this with Hamann's theorem that  $k[p, q][x] \subseteq k[x, y]$  is integral, and get that  $k[p, q] \subseteq k[x, y]$  is integral. Finally, since the integral case has a positive answer, we get that  $f$  is invertible.  $\square$

If we remove the assumption that  $p$  and  $q$  are monic in  $y$ , then we should add the condition that there exist  $\tilde{n}_1 > \tilde{n}_2 > \dots > \tilde{n}_l \geq 0$  and  $a\tilde{n}_2, \dots, a\tilde{n}_l \in k[p, q]$  such that  $y^{\tilde{n}_1} + a\tilde{n}_2 y^{\tilde{n}_2} + \dots + a\tilde{n}_l y^{\tilde{n}_l} \in k(p, q)$  (in order to get that  $y$  is integral over  $k[p, q]$ . So both  $x$  and  $y$  are integral over  $k[p, q]$ , and we arrived at the integral case).

Obviously, the general case result is also true:

**Theorem 5.2.** *Let  $k$  be a field of characteristic zero. Assume  $r_1, \dots, r_m$  are monic in  $x_m$ . If there exist  $n_{1,j} > n_{2,j} > \dots > n_{l,j} \geq 0$  and  $a_{n_{2,j}}, \dots, a_{n_{l,j}} \in k[r_1, \dots, r_m]$  such that for each  $1 \leq j \leq m-1$ ,  $(x_j)^{n_{1,j}} + a_{n_{2,j}} (x_j)^{n_{2,j}} + \dots + a_{n_{l,j}} (x_j)^{n_{l,j}} \in k(r_1, \dots, r_m)$ , then  $f$  is invertible.*

And if we remove the assumption that  $r_1, \dots, r_m$  are monic in  $x_m$ , then we should add the condition that there exist  $n_{1,m} > n_{2,m} > \dots > n_{l,m} \geq 0$  and  $a_{n_{2,m}}, \dots, a_{n_{l,m}} \in k[r_1, \dots, r_m]$  such that  $(x_m)^{n_{1,m}} + a_{n_{2,m}}(x_m)^{n_{2,m}} + \dots + a_{n_{l,m}}(x_m)^{n_{l,m}} \in k(r_1, \dots, r_m)$  (in order to get that  $x_m$  is integral over  $k[r_1, \dots, r_m]$ ). So  $x_1, \dots, x_m$  are integral over  $k[r_1, \dots, r_m]$ , and we arrived at the integral case).

## References

- [1] S. S. Abhyankar, *Expansion techniques in algebraic geometry*, Tata Inst. Fundamental Research, Bombay, 1977.
- [2] M. Atiyah and I. MacDonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Company, 1969.
- [3] H. Bass, E. Connell and D. Wright, *The Jacobian Conjecture: reduction of degree and formal expansion of the inverse*, Bull. Amer. Math. Soc. (New Series) 7 (1982), 287-330.
- [4] H. Bass, Differential structure of Etale extensions of polynomial algebras, *Commutative Algebra* (M. Hochster, C. Huneke and J.D. Sally, eds.), Springer-Verlag, New York, 1989, Proceedings of a Microprogram Held, June 15-July 2, 1987.
- [5] L. A. Campbell, *A condition for a polynomial map to be invertible*, Math. Ann. 205 (1973), 243-248.
- [6] A. van den Essen, *Polynomial automorphisms and the Jacobian conjecture*, Progress in Mathematics 190, Birkhuser Verlag, Basel, 2000.
- [7] E. Formanek, *Observations about the Jacobian Conjecture*, Houston J. of Math. 20, no.3 (1994), 369-380.
- [8] E. Hamann, *Algebraic observations on the Jacobian Conjecture*, J. of Algebra 265, no. 2 (2003), 539-561.
- [9] O. H. Keller, *Ganze Cremona-transformationen*, Monatsh. Math. Phys. 47 (1939), 299-306.
- [10] S. Oda, *The Jacobian problem and the simply-connectedness of  $A^n$  over a field  $k$  of characteristic zero*, Osaka Univ. 1980.
- [11] M. Razar, *Polynomial maps with constant Jacobian*, Israel J. of Math. 32 (1979), 97-106.
- [12] L. H. Rowen, *Graduate algebra: Commutative view*, Graduate Studies in Mathematics, volume 73, Amer. Math. Soc., 2006.
- [13] S. S.-S. Wang, *A Jacobian criterion for separability*, J. of Algebra 65 (1980), 453-494.
- [14] S. S.-S. Wang, *Extension of derivations*, J. of Algebra 69 (1981), 240-246.
- [15] D. Wright, *On the Jacobian Conjecture*, Illinois J. of Math. 15, no. 3 (1981), 423-440.

DEPARTMENT OF MATHEMATICS, BAR-ILAN UNIVERSITY, RAMAT-GAN 52900, ISRAEL.  
*E-mail address:* vered.moskowicz@gmail.com